Novel Approaches to Nearest Neighbors

Random Walks. SEARCH Class.

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Outline

1. Welcome to nearest neighbors!
2. Nearest Neighbors via Random Walks
3. Data Structure Complexity: SEARCH Class

Chapter I

Welcome to Nearest Neighbors!

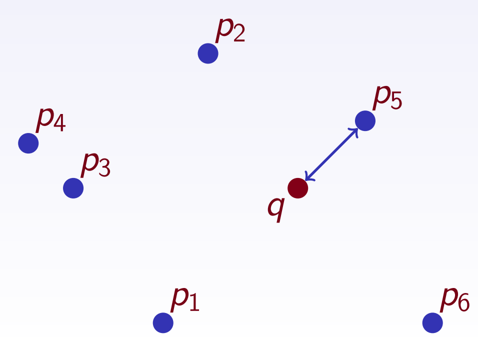
Problem Statement

Search space: object domain $\mathbb{U}$, similarity function σ

Input: database $S = \{p_1, \dots, p_n\} \subseteq \mathbb{U}$

Query: $q \in \mathbb{U}$

Task: find $\operatorname{argmax} \sigma(p_i, q)$



Applications

Content-based retrieval Spelling correction Searching for similar DNA sequences Related pages web search Concept matching kNN classification rule Nearest-neighbor interpolation Near-duplicate detection Plagiarism detection Computing co-occurrence similarity Recommendation systems Personalized news aggregation Behavioral targeting Maximum likelihood decoding MPEG compression

Brief History

- 1908 Voronoi diagram
- 1967 kNN classification rule by Cover and Hart
- 1973 Post-office problem posed by Knuth
- 1997 The paper by Kleinberg, beginning of provable upper/lower bounds
- 2006 Similarity Search book by Zezula, Amato, Dohnal and Batko
- 2008 First International Workshop on Similarity Search. Consider submitting!

Some Nearest Neighbor Solutions

Sphere Rectangle Tree Orchard's Algorithm LAESA k-d-B tree Geometric near-neighbor access tree Excluded middle vantage point forest mvp-tree Fixed-height fixed-queries tree AESA Vantage-point tree R*-tree Burkhard-Keller tree BBD tree Navigating Nets Voronoi tree Balanced aspect ratio tree Metric tree vp^s-tree M-tree Locality-Sensitive Hashing SS-tree R-tree Spatial approximation tree Multi-vantage point tree Bisector tree mb-tree Generalized hyperplane tree Hybrid tree Slim tree Spill Tree Fixed queries tree X-tree k-d tree Balltree Quadtree Octree Post-office tree

Part II

Disorder Inequality

This section represents joint work with Navin Goyal and Hinrich Schütze

Concept of Disorder

Sort all objects in database S by their similarity to p
Let $\text{rank}_p(s)$ be position of object s in this list

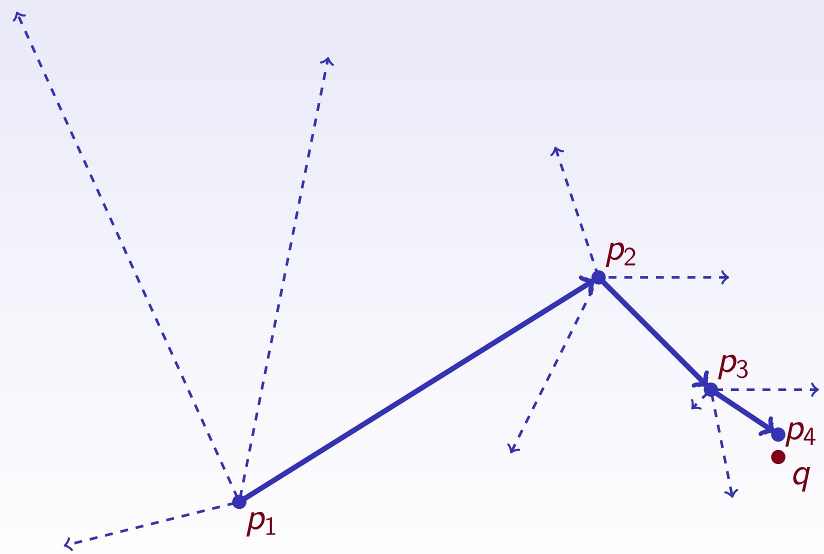
Disorder inequality for some constant D:

$$\forall p, r, s \in \{q\} \cup S : \qquad \text{rank}_r(s) \leq D \cdot (\text{rank}_p(r) + \text{rank}_p(s))$$

Minimal D providing disorder inequality is called disorder constant of a given set

For "regular" sets in d-dimensional Euclidean space $D \approx 2^{d-1}$

Ranwalk Informally (1/2)



Ranwalk Informally (2/2)

Hierarchical greedy navigation:

1. Start at random city p_1
2. Among all airlines choose the one going most closely to q, move there (say, to p_2)
3. Among all railway routes from p_2 choose the one going most closely to q, move there (p_3)
4. Among all bus routes from p_3 choose the one going most closely to q, move there (p_4)
5. Repeat this $\log n$ times and return the final city

Transport system: for level k choose c random arcs to $\frac{n}{2^k}$ neighborhood

Ranwalk Algorithm

Preprocessing:

- For every point p in database we sort all other points by their similarity to p

  Data structure: n lists of $n-1$ points each.

Query processing:

1. Step 0: choose a random point p_0 in the database.
2. From $k=1$ to $k=\log n$ do Step k: Choose $D' := 3D(\log\log n + 1)$ random points from $\min(n, \frac{3Dn}{2^k})$-neighborhood of p_{k-1}. Compute similarities of these points w.r.t. q and set p_k to be the most similar one.
3. If $\text{rank}_{p_{\log n}}(q) > D$ go to step 0, otherwise search the whole D^2-neighborhood of $p_{\log n}$ and return the point most similar to q as the final answer.

Analysis of Ranwalk

Theorem

Assume that database points together with query point $S \cup \{q\}$ satisfy disorder inequality with constant D:

$$rank_x(y) \leq D(rank_z(x) + rank_z(y)).$$

Then Ranwalk algorithm always answers nearest neighbor queries correctly. It uses the following resources:
Preprocessing space: $\mathcal{O}(n^2)$.
Preprocessing time: $\mathcal{O}(n^2 \log n)$.
Expected query time: $\mathcal{O}(D \log n \log\log n + D^2)$.

Arwalk Algorithm

Preprocessing:

- For every point p in database we sort all other points by their similarity to p. For every *level number* k from 1 to $\log n$ we store pointers to $D' = 3D(\log\log n + \log 1/\delta)$ random points within $\min(n, \frac{3Dn}{2^k})$ most similar to p points.

Query processing:

1. Step 0: choose a random point p_0 in the database.
2. From $k = 1$ to $k = \log n$ do Step k: go by p_{k-1} pointers of level k. Compute similarities of these D' points to q and set p_k to be the most similar one.
3. Return $p_{\log n}$.

Analysis of Algorithm

Theorem

Assume that database points together with query point $S \cup \{q\}$ satisfy disorder inequality with constant D:

$$rank_x(y) \leq D(rank_z(x) + rank_z(y)).$$

Then for any probability of error δ Arwalk algorithm answers nearest neighbor query within the following constraints:
Preprocessing space: $\mathcal{O}(nD \log n(\log\log n + \log 1/\delta))$.
Preprocessing time: $\mathcal{O}(n^2 \log n)$.
Query time: $\mathcal{O}(D \log n(\log\log n + \log 1/\delta))$.

Future of Disorder (1/2)

Average disorder. If disorder inequality does not hold for a small fraction of pairs, how should we modify our algorithm?

Improving our algorithms. Is it possible to combine advantages of Ranwalk and Arwalk? Does there exist a deterministic algorithm with sublinear search time utilizing small disorder assumption? E.g., can we use expanders for derandomization?

Future of Disorder (2/2)

Disorder of random sets. Compute disorder values for some modelling examples. For example, consider n random points on d-dimensional sphere

Lower bounds. Is it possible to prove lower bounds on preprocessing and query complexities in some "black-box" model of computation?

Part III

Data Structure Complexity: SEARCH Class

Inclusions with Preprocessing (1/2)

Input

Family $\mathcal{F}$ of subsets of U

Query task

Given a set $f_{new} \subseteq U$ to decide whether $\exists f \in \mathcal{F} : \quad f_{new} \subseteq f$

Constraints

Data storage after preprocessing $poly(|\mathcal{F}| + |U|)$
Time for query processing $poly(|U|)$

Open problem: is there an algorithm satisfying given constraints?

Inclusions with Preprocessing (2/2)

Reformulation in SAT style:

Input

Formula $\mathcal{F}$ in DNF with n variables

Query task

Given an assignment x to evaluate $\mathcal{F}(x)$

Constraints

Data storage after preprocessing $poly(|\mathcal{F}|)$
Time for query processing $poly(n)$

Open problem: is there an algorithm satisfying given constraints?

“NP Analogue” for Search Problems

Every problem in **SEARCH class** is characterized by poly-time computable Turing Machine M:

Input
Strings $x_1, \ldots, x_n$, $\quad |x_i| = m$

Query task
Given string y of length m to answer whether $\exists i : M(x_i, y) = yes$

Tractable problems in SEARCH

Input
Strings $x_1, \ldots, x_n$, $\quad |x_i| = m$

Query task
Given string y of length m to answer whether $\exists i : M(x_i, y) = yes$

Tractable solution
Preprocessing in $poly(m, n)$ space

Query processing in $poly(m, \log n)$ time with RAM access to preprocessed database

Inclusions is in SEARCH. Is it tractable?

Complete problems in SEARCH (1/2)

Program Search problem:

Input
Turing machines $P_1 \ldots, P_n$

Query task
Given string y of length m to answer whether $\exists i : P_i(y) = yes$ after at most m steps

Open problem: is Program Search tractable?

Complete problems in SEARCH (2/2)

Parallel Run problem:

Input
$x_1 \ldots, x_n$

Query task
Given poly-time computable P to answer whether $\exists i : P(x_i) = yes$

Open problem: is Parallel Run tractable?

NN Proofs?

NN-proof system:

- Fix some family of basic statements about points in multidimensional space and some proof system
- Can we compute $poly(|S|)$ statements about points of database S such that for any query q and any real nearest neighbor $p_{NN} \in S$ there is a logarithmic-size proof from precomputed statements that indeed p_{NN} is nearest point is S to q

Do such an NN proof system exist?

Highlights

- Random walk provide logarithmic nearest neighbor search for bounded disorder sets
- SEARCH class: is it tractable?
- Do NN proof systems exist?

Thanks for your attention! Questions?

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